<u>Recall</u> 1) $w : \mathcal{M}_{N,p} \to \mathcal{M}_{N,p}$

$$(E,\alpha,C) \longmapsto (E/C\,,\ \alpha \bmod C\,,\ E[p]/C)$$

2) $\Phi : \overline{\mathcal{M}_N} \to \overline{\mathcal{M}_{N,p}}\ ,\ (E,\alpha)/S \longmapsto (E,\alpha, \ker F_{E/S})$

$w\Phi : (E,\alpha) \longmapsto (E^{(p)}, F_{E/S}\circ\alpha, \ker V_{E/S})$

Let $\pi : \mathcal{M}_{N,p} \to \mathcal{M}_N$ projection.

We obtain: $\pi\circ\Phi = id_{\overline{\mathcal{M}_N}}\ ,\ \pi\circ w\circ\Phi = F_{\overline{\mathcal{M}_N}}$

(absolute Frob. on $\overline{\mathcal{M}_N}$)

Namely given $u : S \to \overline{\mathcal{M}_N}$,

$$\begin{aligned}(\pi\circ w\circ\Phi)(u^*\mathcal{E}, u^*\alpha) &= ((u^*\mathcal{E})^{(p)}, (u^*\alpha)^{(p)}) \\ &= ((u\circ F_S)^*\mathcal{E}, (u\circ F_S)^*\alpha) \\ &= ((F_{\overline{\mathcal{M}_N}}\circ u)^*\mathcal{E}, (F_{\overline{\mathcal{M}_N}}\circ u)^*\alpha)\end{aligned}$$

<u>Thm</u> 1) $\Phi, w\Phi : \overline{\mathcal{M}_N} \to \overline{\mathcal{M}_{N,p}}$ are closed immersions

$\Phi \sqcup w\Phi \leadsto$ bijection on irred comp.

2) $X \subseteq \overline{\mathcal{M}_N}$ irred comp. Then $\Phi(X)$ & $(w\Phi)(X)$

intersect precisely above supersing points,

intersections are ordinary double points.

3) $M_{N,p}$ is regular.

Proof 1) Every section of a separated morphism is a closed immersion. Apply to Φ.

w is automorphism, so $w\Phi$ closed immersion as well.

Bijection on irred comp eventually from prev. lecture.
(left to reader)

2) Ordinary double points

Prop (cf. Stacks 0C4D) $k = \bar{k}$, X/k reduced curve, $x \in X$ closed. Equivalent

1) $\hat{\mathcal{O}}_{X,x} \cong k[[s,t]]/s \cdot t$

2) $\tilde{X} \xrightarrow{\nu} X$ normalization. Then $\operatorname{len}_{\mathcal{O}_{X,x}} (\nu_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X) = 1$.

Def k any X/k curve, $x \in X$ closed, X geom. reduced near x.

x ordinary double point $\overset{=}{\text{def}}$ $\exists\, \bar{x} \in X_{\bar{k}}$, $\bar{x} \longmapsto x$ s.th. $\bar{x}$ satisfies above cond.

Equivalent $\kappa(x)/k$ separable and

$\hat{\mathcal{O}}_{X,x} \cong \kappa[[s,t]]/as^2 + bst + ct^2$ for non-degen quad form.

(Means $\det\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \neq 0$, thus also applies in char $k = 2$.)

Assume we already know $\nu^{-1}(x) = \{x_1, x_2\} \subseteq \widetilde{X}$.
Then condition 2) means that

$$\mathfrak{m}_x (\widehat{\mathcal{O}}_{\widetilde{X},x_1} \times \widehat{\mathcal{O}}_{\widetilde{X},x_2}) = \mathfrak{m}_{x_1} \times \mathfrak{m}_{x_2}.$$

or, equivalently by Nakayama,

$$\begin{array}{ccc} \Omega^1_{X,x} & \xrightarrow{\cong} & \Omega^1_{\widetilde{X},x_1} \oplus \Omega^1_{\widetilde{X},x_2} \\ \| & & \| \qquad\qquad \| \\ \mathfrak{m}_x/\mathfrak{m}_x^2 & \xrightarrow{\cong} & \mathfrak{m}_{x_1}/\mathfrak{m}_{x_1}^2 \oplus \mathfrak{m}_{x_2}/\mathfrak{m}_{x_2}^2 \end{array}.$$

In our case, normalization is

$$\widetilde{X} := \Phi(\overline{M_N}) \amalg w\Phi(\overline{M_N}) \xrightarrow{\nu} X := \overline{M_{N,p}}$$

($\widetilde{X}$ is smooth, so stays normal after $\overline{\mathbb{F}}_p \underset{\mathbb{F}_p}{\otimes} -$)

With $x \in X^{ss}$, $x_1 = \Phi(x_1)$, $x_2 = w\Phi(x)$:

$\pi \circ \nu = \mathrm{id}_{\overline{M_N}} \amalg F_{\overline{M_N}}$, so $\Omega^1_{X_1,x_1} \oplus 0 \subseteq$ Image of $\Omega^1_{X,x}$.

Similarly $\pi \circ \langle p\rangle^{-1} w \circ \nu = \langle p\rangle^{-1} F_{\overline{M_N}} \amalg \mathrm{id}_{\overline{M_N}}$,

so similarly $0 \oplus \Omega^1_{X_2,x_2} \subseteq$ Image of $\Omega^1_{X,x}$ $\square$ 2)

§ Regularity of $\mathcal{M}_{N,p}$

$N \geq 3$, $(p,N)=1$

Aim: $\mathcal{M}_{N,p}$ is regular in all points of $\overline{\mathcal{M}}_{N,p}^{ss}$

(Reminder: $\mathcal{M}_{N,p} \setminus \overline{\mathcal{M}}_{N,p}^{ss} \longrightarrow \operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ is smooth, local structure as good as it can be.)

Reduction: Regularity is étale local. Let $\mathbb{Z}_{(p)} - W$ be unramified extn of DVRs, $W/p = \mathbb{F}_q$, s.th. all points of $\overline{\mathcal{M}}_{N,p}^{ss}$ $\mathbb{F}_q$-rational.

(Rmk: For any q, may find such a W as localization of $\mathbb{Z}[\zeta_{q-1}]$.)

Enlarging $\mathbb{F}_q$, may suppose tangent directions of $\mathbb{F}_q \otimes \overline{\mathcal{M}}_{N,p}$ in all sup. sing. points rational, i.e.

$$\hat{\mathcal{O}}_{\mathbb{F}_q \otimes \overline{\mathcal{M}}_{N,p}, x} \cong \mathbb{F}_q[[s,t]]/s \cdot t \quad \forall \text{ sup. sing. } x.$$

(This is actually automatic b/c we have a section $\Phi : \overline{\mathcal{M}}_N \to \overline{\mathcal{M}}_{N,p}$, q need not be enlarged.)

Step ① Show $\hat{\mathcal{O}}_{W\otimes M_{N,p},x} \cong W[\![s,t]\!]/st-p^m$

for some m. (pure commutative algebra)

Step ② Use moduli property to see $m=1$.

Done b/c $W[\![s,t]\!]/_{s\cdot t-p}$ is regular!

(Maximal ideal is generated by two elements:

$(s,t,p) = (s,t,s\cdot t) = (s,t)$)

Ad Step 1 Prop 5.3 Deligne - Rapoport

R noetherian complete local, $C \to S = \operatorname{Spec} R$ curve,

i.e. proper, flat, rel. 1-dimensional.

$x \in \underbrace{C(s)}_{\text{closed point}}$ s.th. $\hat{\mathcal{O}}_{C(s),x} \cong \kappa(s)[\![u,v]\!]/_{u\cdot v}$.

Then $\exists\ \lambda \in R$ s.th.

$\hat{\mathcal{O}}_{C,x} \cong R[\![u,v]\!]/_{u\cdot v-\lambda}$.

In our situation, the base W is a DVR, so we can

give an easy direct argument.

__To see__ $\hat{\mathcal{O}}_{\mathcal{M}_{N,p},x} \cong W[[s,t]]/_{st - p^m}$ for some $m \geq 1$.

__Proof__ Have a surjection

$$\begin{tikzcd}
\hat{\mathcal{O}}_{\mathcal{M}_{N,p},x} \arrow[r] & \hat{\mathcal{O}}_{\overline{\mathcal{M}}_{N,p},x} \cong \mathbb{F}_q[[s,t]]/_{s\cdot t} \\
 & W[[s,t]] \arrow[u, two heads] \arrow[ul, dashed]
\end{tikzcd}$$

- Pick any lift.
- By Nakayama surjective
- $I :=$ kernel
- $(p, I) = (p, s\cdot t)$

$\rightsquigarrow$ $\exists$ elements of form $f = s\cdot t + p\cdot g(s,t) \in I$,

pick any.

__Claim__ f generates I.

__Proof__ Given $h \in I$, may write $h = h_1 \cdot p + h_2 \cdot st$

since $I \subseteq (p, s\cdot t)$.

Then $h - h_2 \cdot f \in pW \cap I$.

Since $\mathcal{M}_{N,p}$ flat over $\mathbb{Z}[\frac{1}{N}]$, i.e. $\mathcal{O}_{\mathcal{M}_{N,p}}$ p-torsion free

$\frac{1}{p}(h - h_2 f) \in I$.

Iterate, use completeness of $W[[s,t]]$ $\square$

Claim After coord. trafo, f has claimed form $s \cdot t - p^m$.

Proof Write $f = c + s t + p(s h(s) + t k(s,t))$

Then $f(s - p \cdot k(s,t), t - p h(s))$

$$= c + st - pt k(s,t) - p s h(s) + p^2 k(s,t) h(s)$$
$$+ p s h(s) + p t k(s,t)$$
$$= c + st + p^2 (s h_2(s) + t k_2(s,t))$$

Iteration + completeness $\Longrightarrow$ $f = c + st$

Finally, use $s \longmapsto c/p^{val(c)} \in W^\times$ $\square$

Ad Step 2)

Claim There exist non-units $f, g \in W[[s,t]]/st - p^m$ with $f \cdot g = p$ only when $m = 1$.

Proof Checked directly from explicit description:

Every $f \in W[[s,t]]/s \cdot t - p^m$ unique expression as

$$\text{const} + \sum_{i=1}^{\infty} a_i s^i + \sum_{i=1}^{\infty} b_i t^i \qquad a_i, b_i \in W.$$

$\square$

Where do f, g w/ $f \cdot g = p$ in $\hat{\mathcal{O}}_{\mathcal{M}_{N,p}, x}$ come from?

Over $\mathcal{M}_{N,p}$, can consider universal isogenies

$$\mathcal{E} \xrightarrow[\text{quotient } \pi]{} \mathcal{E}/\mathcal{C} \xrightarrow[\pi^{-1}\cdot p \text{ dual isogeny.}]{} \mathcal{E}, \qquad \mathcal{C} \subseteq \mathcal{E} \text{ universal order-}p \text{ group}$$

Given $\mathrm{Lie}\, \mathcal{E} \xrightarrow{f} \mathrm{Lie}\,(\mathcal{E}/\mathcal{C}) \xrightarrow{g} \mathrm{Lie}\, \mathcal{E}$

with $g \circ f = \mathrm{Lie}\,[p] = p$.

f (resp. g) sections of $(\mathrm{Lie}\, \mathcal{E})^{-1} \otimes (\mathrm{Lie}\, \mathcal{E}/\mathcal{C})$

(resp. $(\mathrm{Lie}\, \mathcal{E}/\mathcal{C})^{-1} \otimes (\mathrm{Lie}\, \mathcal{E})$),

but we may locally trivialize those line bundles to produce the desired $f, g \in \mathcal{O}_{\mathcal{M}_{N,p}, x}$.

<u>Observe</u> $f|_{\Phi(\overline{\mathcal{M}}_N)} = 0$, $f|_{w\Phi(\overline{\mathcal{M}}_N^{\mathrm{ord}})}$ invertible

$g|_{w\Phi(\overline{\mathcal{M}}_N)} = 0$, $g|_{\Phi(\overline{\mathcal{M}}_N^{\mathrm{ord}})}$ invertible

In partic, neither is a unit in supersingular pts. $\Rightarrow$ Regularity !!

Variant of Step 2 $\hat{\mathcal{O}}_{\mathcal{M}_{N,p}, x} \cong W[[s,t]]/_{s.t - p^m}$

implies there is $\operatorname{Spec} W/p^m \longrightarrow \mathcal{M}_{N,p}$ with image x. (Send $s,t \mapsto 0$.)

In other words, $\exists\ (E, \alpha, C)/_{W/p^m}$ s.th.

E mod p is supersingular, in particular

$$\overline{\mathbb{F}}_q \underset{\mathbb{F}_q}{\otimes} C \cong \alpha_p .$$

Now apply the following classification of order-p groups:

<u>Thm</u> (Oort-Tate) R complete, noetherian, local, char $R/\mathfrak{m} = p$.

Then

$$\left\{ G/R \ \begin{array}{l} \text{order } p \text{ comm. fin.} \\ \text{loc. free grp sch} \end{array} \right\}/_{\cong} \simeq \left\{ \begin{array}{l} a, b \in R \text{ s.th.} \\ ab = p \end{array} \right\}/_{\sim}$$

where $(a,b) \sim (u^{p-1}a, u^{1-p}b)$ $u \in R^\times$.

Under this bijection, $\alpha_p/_k \leftrightarrow a = b = 0$.

$(\mu_p \longleftrightarrow a$ unit, $b = 0$, $\mathbb{Z}/p \longleftrightarrow a = 0$, b unit$)$

But there are no $a, b \in W/p^2$, $a \equiv b \equiv 0 \ (p)$

s.th. $a \cdot b = p$.

$\Rightarrow$ C cannot deform over W/p^2

$\Rightarrow$ $m = 1$ as desired. $\blacksquare$